JOURNAL OF APPROXIMATION THEORY **70**, 358–374 (1992)

On Discrete Simplex Splines and Subdivision

MARIAN NEAMTU*

Department of Applied Mathematics, University of Twente,
P. O. Box 217, 7500 AE Enschede, The Netherlands

Communicated by Charles A. Micchelli

Received December 19, 1990; revised June 17, 1991

Discrete analogoues of multivariate simplex splines are introduced. Their study yields a subdivision scheme for simplex splines. © 1992 Academic Press, Inc.

1. INTRODUCTION

A variety of techniques to facilitate fast graphical display of curves and surfaces for interactive CAGD purposes have been developed, in the past years. Many of these are commonly referred to as *subdivision algorithms* [2].

In this paper we propose a method for subdividing *simplex splines*, i.e., splines that can be defined over *non-regular* grid partitions. There is a relatively exhaustive theory of multivariate simplex splines (or shortly *B*-splines) [6]. Due to some favorable properties, similar to those in the univariate case, the *B*-splines are suitable, in principle, for application, e.g., in the *Finite Element Method* or in *Computer Aided Geometric Design*. On the other hand, their computational properties seem to be very restrictive compared to other available methods. In spite of the existence of a number of *recurrence relations* to facilitate the numerical manipulation of *B*-splines, these are often computationally expensive in sharp contrast with the univariate case. Therefore, there naturally arises a need for a subdivision-like algorithm for simplex *B*-splines. Some basic ideas on subdivision of *B*-splines have been described in [1, 12, 14, 15], all based on a geometric interpretation of *B*-splines. Here, we suggest a different approach. It is based on the notion of *discrete simplex splines*, which is proven to be an extension of the ideas underlying the definitions of *discrete cube splines*, also called *box splines*, and *discrete cone splines* [3, 7]. The notion of the

* Supported by the Netherlands Foundation for Mathematics (SMC) with financial aid from the Netherlands Organization for the Advancement of Research (NWO).

0021-9045/92 $5.00

discrete B-splines will turn out to be useful for the purpose of subdividing their continuous counterparts.

The paper consists of the following parts. We start with a brief review of basic facts about discrete cube splines and we give some extensions needed for later purposes. Then the discrete B-splines are defined and some of their properties are derived. Sections 4 and 5 deal with an algorithm for subdividing B-splines. We finish the paper with a discussion.

2. Discrete Cube and Cone Splines

Let $X := \{x^1, ..., x^n\}$, $x^i \in \mathbb{R}^s \setminus \{0\}$, and $n \geq s$. In the following we assume that the linear span of X, denoted by $\langle X \rangle$, is $\mathbb{R}^s$. The cube spline $B(\cdot \mid X)$ and the cone spline $T(\cdot \mid X)$ are defined by, respectively,

$$\int_{\mathbb{R}^s} B(x \mid X)\, f(x)\, dx = \int_{[0,1]^n} f(Xv)\, dv, \tag{2.1}$$

and

$$\int_{\mathbb{R}^s} T(x \mid X)\, f(x)\, dx = \int_{\mathbb{R}^n_+} f(Xv)\, dv, \tag{2.2}$$

where $Xv := x^1 v_1 + \cdots + x^n v_n$, which must hold for any continuous and locally supported function f. The latter definition makes sense if $0 \notin [X]$, $[X]$ the convex hull of X.

Next, let $H := \operatorname{diag}(h_1, ..., h_n)$, $h_i =: p_i^{-1}$, $p_i \in \mathbb{N}$, be a diagonal scaling matrix. The discrete version $b_H(\cdot \mid X)$ of the cube spline is defined as [3, 7]

$$\sum_{x \in \mathscr{L}_H(X)} b_H(x \mid X)\, f(x) = \det(H) \sum_{\substack{v \in \mathbb{Z}^n \\ 0 \leq v_i < p_i}} f(XHv), \tag{2.3}$$

which should hold for any discrete function f vanishing in all but a finite number of points in $\mathscr{L}_H(X)$. Here $\mathscr{L}_H(X) := \{x \in \mathbb{R}^s \mid x = XHv,\ v \in \mathbb{Z}^n\}$, and $XHv := x^1 h_1 v_1 + \cdots + x^n h_n v_n$. Similarly, the discrete cone spline $t_H(\cdot \mid X)$ can be defined by the relation [7]

$$\sum_{x \in \mathscr{L}_H(X)} t_H(x \mid X)\, f(x) = \det(H) \sum_{v \in \mathbb{Z}^n_+} f(XHv),$$

which is, under the condition $0 \notin [X]$, to be satisfied for all f vanishing in all but a finite number of points in $\mathscr{L}_H(X)$.

The properties and structure of discrete cube splines and cone splines were intensively studied, e.g., in [9–11]. It has been shown in [3, 7], that the discrete cube splines converge in the pointwise sense to the continuous cube splines, if $p_i \to \infty$. It is important to remark that the conditions on X and H under which the convergence was proved are only sufficient but not necessary. As a consequence, the results in [3, 7] do not imply that the discrete cube spline converges to its continuous counterpart for all $X \subset \mathbb{Z}^s$. It turns out, however, that the convergence is indeed guaranteed, as stated in the next lemma.

We first need the following notation. Let $X \subset \mathbb{Z}^s$. We call a set $Y_H = \{y_H^1, ..., y_H^s\} \subset \mathbb{R}^s$ a *basis* for $\mathscr{L}_H(X)$ if $\langle Y_H \rangle = \mathbb{R}^s$ and $\mathscr{L}(Y_H) := \{x \in \mathbb{R}^s \mid x = Y_H \mu, \quad \mu \in \mathbb{Z}^s\} = \mathscr{L}_H(X)$, where $Y_H \mu := y_H^1 \mu_1 + \cdots + y_H^s \mu_s$. Note that in general the basis of $\mathscr{L}_H(X)$ is not uniquely determined. All bases of $\mathscr{L}_H(X)$ have the same (nonzero) value of $|\det(Y_H)|$, however. For instance, if $s = 2$, $X = \{(1, 0), (0, 1), (1, 1)\}$ and $H = \operatorname{diag}(1, 1, 2/3)$, then both sets $Y_H^1 = \{(1/3, 0), (0, 1/3)\}$ and $Y_H^2 = \{(1/3, 0), (1/3, 1/3)\}$ form a basis of $\mathscr{L}_H(X)$ and

$$|\det(Y_H^1)| = |\det(Y_H^2)|.$$

LEMMA 2.1. *Let $X \subset \mathbb{Z}^s$ and let Y_H be a basis for $\mathscr{L}_H(X)$. Then there is a constant γ depending on X and s such that*

$$|B(x \mid X) - |\det(Y_H)|^{-1} b_H(x \mid X)| \leqslant \gamma \|H\|,$$

for all $x \in \mathscr{L}_H(X)$, where $\|H\| := \max\{h_1, ..., h_n\}$ and $\mathscr{L}(Y_H) := \{x \in \mathbb{R}^s \mid x = Y_H v,\ v \in \mathbb{Z}^n\}$.

We omit the proof here, since it can be done following the same lines as the proof of the convergence result in [7]. Some remarks on the lemma are in order:

— If $Y_H \subset XH := \{x^1 h_1, ..., x^n h_n\}$, then exactly the result in [7] is reproduced.

— If $X \not\subset \mathbb{Z}^s$, then the necessary and sufficient condition for the convergence of the discrete cube spline is that there is a set $Y_H = \{y_H^1, ..., y_H^s\}$ with $\det(Y_H) \neq 0$ forming a basis for $\mathscr{L}_H(X)$, or equivalently, there must exist a set $Y = \{y^1, ..., y^s\}$ which forms a basis for $\mathscr{L}(X)$.

For the rest of this paper we assume that $X \subset \mathbb{Z}^s$ and $H = \operatorname{diag}(h, ..., h)$, where $h^{-1} =: p \in \mathbb{N}$. Accordingly, we modify the notational convention: instead of H we employ the lower case h.

A simple consequence of the definitions (2.1), (2.3) is the following lemma.

LEMMA 2.2. *Let $h_i^{-1} = p_i \in \mathbb{N}$, $i = 1, 2$. Then*

$$b_{h_1 h_2}(x \mid X) = h_2^n \sum_{0 \leqslant v_i < p_2} \bar{b}_{h_1}(x - X h_1 h_2 v \mid X), \tag{2.4}$$

for all $x \in \mathscr{L}_{h_1 h_2}(X)$, *where*

$$\bar{b}_{h_1}(x \mid X) := \begin{cases} b_{h_1}(x \mid X), & \text{if } \; x \in \mathscr{L}_{h_1}(X) \\ 0, & \text{otherwise.} \end{cases}$$

Formula (2.4) suggests a recipe for subdividing any cube spline with $X \subset \mathbb{Z}^s$. There is, however, a computationally more convenient way of calculating $b_{h_1 h_2}$, which can be considered as a generalization of the *line average algorithm* [3, 7, 8].

COROLLARY 2.1. $b_{h_1 h_2}$ *can be calculated for all* $x \in \mathscr{L}_{h_1 h_2}(X)$ *by the following algorithm.*

1. $b^0_{h_1}(x \mid X) := \bar{b}_{h_1}(x \mid X)$,
2. $b^i_{h_1}(x \mid X) := h_2 \sum_{j=0}^{p_2-1} b^{i-1}_{h_1}(x - jh_1 h_2 x^i \mid X)$, $i = 1, ..., n$,
3. $b_{h_1 h_2}(x \mid X) = b^n_{h_1}(x \mid X)$.

Since in the rest of the paper we will be concerned with discrete cone splines rather than with cube splines, we remark that results similar to those in Lemmas 2.1, 2.2, and Corollary 2.1 can be obtained without difficulties also for discrete cone splines.

3. DISCRETE B-SPLINES

For the remainder of the paper we reserve the symbol X for the set $X := \{x^0, ..., x^n\}$, $n \geqslant s$, where x^i are points in $\mathbb{R}^s$, sometimes also called *knots*. We recall the distributional definition of the multivariate B-spline $M(\cdot \mid X)$ associated with the knot-set X, which requires the relation

$$\int_{\mathbb{R}^s} M(x \mid X) f(x)\, dx = n! \int_{S^n} f(v_0 x^0 + \cdots + v_n x^n)\, dv_1 \cdots dv_n, \tag{3.1}$$

to be satisfied for all $f \in C(\mathbb{R}^s)$, where $S^n := \{(v_0, ..., v_n) \mid \sum_{i=0}^n v_i = 1,\ v_i \geqslant 0\}$ is the *standard n-simplex.*

The next well known identity, first derived by Dahmen [4, 5], relates B-splines to cone splines,

$$M(x \mid X) = n! \sum_{i=0}^{n} (-1)^i\, T(x - x^i \mid X^i), \tag{3.2}$$

where $X^i := \{x^i - x^0, ..., x^i - x^{i-1}, x^{i+1} - x^i, ..., x^n - x^i\}$, $i = 0, ..., n$, which is valid under the assumption that

$$\text{all knots in } X \text{ are distinct and } 0 \notin [X^i],\ i = 0, ..., n. \tag{3.3}$$

Note, that if all knots in X are distinct, then the condition $0 \notin [X^i]$, $i = 0, ..., n$, is actually a restriction on the ordering of X rather than on the values of the knots. In case $\mathrm{vol}_s[X] = 0$, the s-dimensional volume of $[X]$, the above identity is meaningful only in the distributional sense. From now on we assume $\mathrm{vol}_s[X] \neq 0$, however. Since we will give a discrete version of (3.2), we include a short proof, different from the one in [4, 5].

Proof of (3.2). Define the sets

$$C_0^n := \left\{ (v_0, ..., v_n) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i=0}^{n} v_i = 1,\ v_1 \geqslant 0, ..., v_n \geqslant 0 \right\},$$

$$C_i^n := \left\{ (v_0, ..., v_n) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i=0}^{n} v_i = 1,\ v_0 < 0, ..., v_{i-1} < 0, \right.$$

$$\left. v_{i+1} \geqslant 0, ..., v_n \geqslant 0 \right\}, \qquad i = 1, ..., n-1, \tag{3.4}$$

$$C_n^n := \left\{ (v_0, ..., v_n) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i=0}^{n} v_i = 1,\ v_0 < 0, ..., v_{n-1} < 0 \right\}.$$

It can be immediately seen that

$$S^n = C_0^n \backslash C_1^n \backslash \cdots \backslash C_n^n, \tag{3.5}$$

where the difference operation "\" is defined such that whenever A, B, C are three sets, then $A \backslash B \backslash C := A \backslash (B \backslash C)$. Applying (3.5) to (3.1) yields

$$\begin{aligned}
&\int_{\mathbb{R}^s} M(x \mid X)\, f(x)\, dx \\
&\quad = n! \int_{S^n} f(v_0 x^0 + \cdots + v_n x^n)\, dv_1 \cdots dv_n \\
&\quad = n! \int_{C_0^n} f(v_0 x^0 + \cdots + v_n x^n)\, dv_1 \cdots dv_n - \cdots \\
&\qquad + (-1)^n\, n! \int_{C_n^n} f(v_0 x^0 + \cdots + v_n x^n)\, dv_1 \cdots dv_n.
\end{aligned}$$

Employing the definition of the sets C_i^n, (2.2), (3.3) and an elementary calculation readily finishes the proof. ∎

In the following we give a definition of the multivariate discrete B-splines. To that end let $S^{n,h}$ be the *discrete standard n-simplex*,

$$S^{n,h} := \left\{ (v_0, ..., v_n) \in \mathbb{Z}_+^{n+1} \,\middle|\, \sum_{i=0}^{n} h v_i = 1 \right\},$$

and

$$\tilde{\mathscr{L}}_h(X) := \left\{ x \in \mathbb{R}^s \mid x = Xhv,\ v \in \mathbb{Z}^{n+1},\ \sum_{i=0}^{n} hv_i = 0 \right\},$$

$$\mathscr{S}_h(X) := \left\{ x \in \mathbb{R}^s \mid x = Xhv,\ v \in \mathbb{Z}^{n+1},\ \sum_{i=0}^{n} hv_i = 1 \right\}.$$

DEFINITION 3.1. The multivariate discrete simplex B-spline $m_h(\cdot \mid X)$ is a function defined on $\mathscr{S}_h(X)$ satisfying

$$\sum_{x \in \mathscr{S}_h(X)} m_h(x \mid X)\, f(x) = (\# S^{n,h})^{-1} \sum_{v \in S^{n,h}} f(Xhv), \tag{3.6}$$

for all locally supported discrete functions f. $\# S^{n,h}$ denotes the cardinality of $S^{n,h}$, which is

$$\# S^{n,h} = \binom{n + h^{-1}}{n} = \binom{n+p}{n}. \tag{3.7}$$

The following identity which is a consequence of (3.6) could be used to give an equivalent definition of discrete B-splines,

$$m_h(x \mid X) = (\# S^{n,h})^{-1}\ \# \{ v \in S^{n,h} \mid Xhv = x \}, \qquad x \in \mathscr{S}_h(X). \tag{3.8}$$

This implies that $m_h(\cdot \mid X)$ is a nonnegative function with local support contained in $[X] \cap \mathscr{S}_h(X)$. Moreover, it is normalized such that

$$\sum_{x \in \mathscr{S}_h(X)} m_h(x \mid X) = 1.$$

From now on we assume $X \subset \mathbb{Z}^s$. The next identity is a discrete analog of (3.2).

THEOREM 3.1. *Let X satisfy* (3.3) *and let* $\xi^0 := 0$, $\xi^i := x^i - x^0 + \cdots + x^i - x^{i-1}$, $i = 1, ..., n$. *Then*

$$m_h(x \mid X) = l(n, h) \sum_{i=0}^{n} (-1)^i\, t_h(x - x^i - h\xi^i \mid X^i), \tag{3.9}$$

for all $x \in \mathscr{S}_h(X)$, *where* $l(n, h) := h^{-n}(\# S^{n,h})^{-1}$.

Proof. Note first, that both sides of (3.9) are defined on the same lattice $\mathscr{S}_h(X)$. To see this, observe that both cone splines $t_h(\cdot \mid X^i)$ and $t_h(\cdot - h\xi^i \mid X^i)$, $i = 0, ..., n$, are defined on $\mathscr{L}_h(X^i)$, since $h\xi^i \in \mathscr{L}_h(X^i)$. This implies that $t_h(\cdot - x^i - h\xi^i \mid X^i)$ is defined on $\mathscr{S}_h(X)$. In analogy to (3.4) we define sets $C_i^{n,h}$ as

$$C_0^{n,h} := \left\{ (v_0, ..., v_n) \in \mathbb{Z}^{n+1} \,\middle|\, \sum_{i=0}^{n} hv_i = 1,\ v_1 \geqslant 0, ..., v_n \geqslant 0 \right\},$$

$$C_i^{n,h} := \left\{ (v_0, ..., v_n) \in \mathbb{Z}^{n+1} \,\middle|\, \sum_{i=0}^{n} hv_i = 1,\ v_0 < 0, ..., v_{i-1}, < 0, \right.$$

$$\left. v_{i+1} \geqslant 0, ..., v_n \geqslant 0 \right\}, \qquad i = 1, ..., n-1,$$

$$C_n^{n,h} := \left\{ (v_0, ..., v_n) \in \mathbb{Z}^{n+1} \,\middle|\, \sum_{i=0}^{n} hv_i = 1,\ v_0 < 0, ..., v_{n-1} < 0 \right\}.$$

Since $S^{n,h} = C_0^{n,h} \setminus C_1^{n,h} \setminus \cdots \setminus C_n^{n,h}$, we can write

$$\begin{aligned}
\sum_{x \in \mathscr{S}_h(X)} & m_h(x \mid X) f(x) \\
&= (\# S^{n,h})^{-1} \sum_{v \in S^{n,h}} f(Xhv) \\
&= (\# S^{n,h})^{-1} \sum_{i=0}^{n} (-1)^i \sum_{v \in C_i^{n,h}} f(Xhv) \\
&= (\# S^{n,h})^{-1} \sum_{i=0}^{n} (-1)^i \sum_{v \in C_i^{n,h}} f(x^i + h\xi^i - h(v_0+1)(x^i - x^0) - \cdots \\
&\quad - h(v_{i-1}+1)(x^i - x^{i-1}) + hv_{i+1}(x^{i+1} - x^i) + \cdots + hv_n(x^n - x^i)).
\end{aligned}$$

By the following transformation of variables,

$$\lambda_i := (-(v_0+1), ..., -(v_{i-1}+1), v_{i+1}, ..., v_n), \qquad i = 0, ..., n,$$

the last term of the above equation equals

$$\begin{aligned}
(\# S^{n,h})^{-1} & \sum_{i=0}^{n} (-1)^i \sum_{\lambda_i \in \mathbb{Z}_+^n} f(x^i + h\xi^i + X^i h\lambda_i) \\
&= (\# S^{n,h})^{-1} \sum_{i=0}^{n} (-1)^i h^{-n} \sum_{x \in \mathscr{S}_h(X)} t_h(x - x^i - h\xi^i \mid X^i) f(x) \\
&= \sum_{x \in \mathscr{S}_h(X)} \left(l(n, h) \sum_{i=0}^{n} (-1)^i t_h(x - x^i - h\xi^i \mid X^i) \right) f(x),
\end{aligned}$$

which finishes the proof. ∎

The following assertion concerns the generating function of $m_h(\cdot\,|\,X)$. Observe first, that if $z=(z_1, ..., z_s)\in\mathbb{C}^s$, $z^\alpha := z_1^{\alpha_1}\cdots z_s^{\alpha_s}$ for $\alpha=(\alpha_1, ..., \alpha_s)$ and $\bar{X} := \{x^1, ..., x^n\}$, then the *generating function* $\tilde{t}_h(z\,|\,\bar{X})$ of $t_h(x\,|\,\bar{X})$ is

$$\tilde{t}_h(z\,|\,\bar{X}) := h^s \sum_{x\in\mathscr{L}_h(\bar{X})} z^x t_h(x\,|\,\bar{X}) = \frac{h^n}{\prod_{j=1}^n (1-z^{hx^j})}. \tag{3.10}$$

This has been shown in [11] for the case $h=1$.

LEMMA 3.1. *Let X satisfy* (3.3). *Then the generating function $\tilde{m}_h$ of m_h is a divided difference of z^{p+n}, namely*

$$\tilde{m}_h(z\,|\,X) := h^s \sum_{x\in\mathscr{S}_h(X)} z^x m_h(x\,|\,X) = \binom{p+n}{n}^{-1} [z^{hx^0}, ..., z^{hx^n}](\cdot)^{p+n}.$$

Proof. From (3.10) it follows that the generating function of $t_h(x-x^i-h\xi^i\,|\,X^i)$ is

$$\frac{h^n z^{x^i+h\xi^i}}{\prod_{j=0,\,j\neq i}^n (1-z^{h|x^i-x^j|})},$$

where

$$|x^i-x^j| := \begin{cases} x^i-x^j, & \text{if } j<i \\ x^j-x^i, & \text{if } j>i. \end{cases}$$

Hence, based on the identity (3.9) we obtain

$$\begin{aligned}
\tilde{m}_h(z\,|\,X) &= h^s \sum_{x\in\mathscr{S}_h(X)} z^x m_h(x\,|\,X) \\
&= l(n,h) \sum_{i=0}^n (-1)^i \frac{h^n z^{x^i+h\xi^i}}{\prod_{j=0,\,j\neq i}^n (1-z^{h|x^i-x^j|})} \\
&= (\# S^{n,h})^{-1} \sum_{i=0}^n \frac{z^{x^i}}{\prod_{j=0,\,j\neq i}^n (1-z^{h(x^j-x^i)})} \\
&= (\# S^{n,h})^{-1} \sum_{i=0}^n \frac{z^{hx^i(h^{-1}+n)}}{\prod_{j=0,\,j\neq i}^n (z^{hx^i}-z^{hx^j})}.
\end{aligned}$$

Recalling the familiar identity for the univariate divided differences,

$$[a_0, ..., a_n]\, f(\cdot) = \sum_{i=0}^n \frac{f(a_i)}{\prod_{j=0,\,j\neq i}^n (a_i-a_j)}, \qquad a_i\in\mathbb{R},$$

and (3.7) readily finishes the proof. ∎

Note, that this lemma brings in mind the well known fact that the *Fourier transform* of a (continuous) B-spline is a divided difference of an exponential function (cf. [6]). There is another property of continuous B-splines, which is retained in the discrete case. Let $\lambda \in \mathbb{Z}^s$ be such that for every $k \in \mathscr{L}_h(\bar{X}\lambda)$,

$$\#(\{x \in \mathscr{L}_h(\bar{X}) \mid \lambda x = k\} \cap \{\bar{X}v, v \in \mathbb{R}^n_+\}) < \infty,$$

then

$$\mathscr{R}_\lambda(t_h(\cdot \mid \bar{X}))(k) := \sum_{\substack{\lambda x = k \\ x \in \mathscr{L}_h(\bar{X})}} t_h(x \mid \bar{X}) = t_h(k \mid \bar{X}\lambda).$$

That is, the *discrete Radon transform* of the discrete multivariate cone spline is a *denumerant*, i.e., a univariate discrete cone spline. The proof of this fact, in a somewhat modified form, can be found in [11]. This result can easily be extended to give

$$t_h(k - x^i\lambda - h\xi^i\lambda \mid X^i\lambda) = \sum_{\substack{\lambda x = k \\ x \in \mathscr{S}_h(X)}} t_h(x - x^i - h\xi^i \mid X^i), \qquad k \in \mathscr{S}_h(X\lambda).$$

Hence, under the assumption (3.3), Theorem 3.1 leads to

$$\begin{aligned}
\sum_{\substack{\lambda x = k \\ x \in \mathscr{S}_h(X)}} m_h(x \mid X) &= \sum_{\substack{\lambda x = k \\ x \in \mathscr{S}_h(X)}} l(n, h) \sum_{i=0}^{n} (-1)^i t_h(x - x^i - h\xi^i \mid X^i) \\
&= l(n, h) \sum_{i=0}^{n} (-1)^i \sum_{\substack{\lambda x = k \\ x \in \mathscr{S}_h(X)}} t_h(x - x^i - h\xi^i \mid X^i) \\
&= l(n, h) \sum_{i=0}^{n} (-1)^i t_h(k - x^i\lambda - h\xi^i \mid X^i\lambda) \\
&= m_h(k \mid X\lambda),
\end{aligned}$$

which proves

LEMMA 3.2. *$\mathscr{R}_\lambda(m_h(\cdot \mid X))(k) = m_k(k \mid X\lambda)$, that is, the discrete Radon transform of the multivariate B-spline is a univariate discrete B-spline.*

Notice, that the same is true in the continuous case [13, 6]. Next we give a discrete analogue of another interesting property of the continuous B-splines, namely we show that the restriction of an $(s+1)$-variate discrete cone spline to an s-dimensional hyperplane is an s-variate discrete B-spline. In particular, let $\tilde{x}^i := (x^i, 1)$, $\tilde{X} := \{\tilde{x}^0, ..., \tilde{x}^n\}$, $\tilde{x} := (x, 1)$, then

LEMMA 3.3.

$$m_h(x \mid X) = l(n, h)\, t_h(\tilde{x} \mid \tilde{X}), \tag{3.11}$$

for all $x \in \mathscr{S}_h(X)$.

Proof. To see that the right hand side of (3.11) is defined for all $x \in \mathscr{S}_h(X)$, it is sufficient to observe that the restriction of $\mathscr{L}_h(\tilde{X})$ to the hyperplane $x_{s+1} = 1$ equals

$$\left\{(x, 1) \mid x = Xhv,\ \sum_{i=0}^{n} hv_i = 1,\ v \in \mathbb{Z}^{n+1}\right\} = \{(x, 1) \mid x \in \mathscr{S}_h(X)\}.$$

The proof of (3.11) is based on the identity (3.8). Namely,

$$\begin{aligned} l(n, h)\, t_h(\tilde{x} \mid \tilde{X}) &= l(n, h)\, h^n\, \#\{v \in \mathbb{Z}_+^{n+1} \mid \tilde{X}hv = \tilde{x}\} \\ &= (\# S^{n,h})^{-1}\, \#\left\{v \in \mathbb{Z}_+^{n+1} \mid Xhv = x,\ \sum_{i=0}^{n} hv_i = 1\right\} \\ &= (\# S^{n,h})^{-1}\, \#\{v \in S^{n,h} \mid Xhv = x\} = m_h(x \mid X). \quad \blacksquare \end{aligned}$$

We now prove the following recurrence relation relating higher-order discrete B-splines to lower-order ones.

LEMMA 3.4. *If* $n > s$, *then for every* $j \in \{0, ..., n\}$,

$$m_h(x \mid X) = \binom{p+n}{n}^{-1} \sum_{i=0}^{p-1} \binom{p+n-i-1}{n-1} m_{h_i}(x - ihx^j \mid X \backslash \{x^j\}),$$

$$h_i := (p-i)^{-1}.$$

Proof. On account of (3.8), we obtain

$$\begin{aligned} &m_h(x \mid X) \\ &\quad = \binom{p+n}{n}^{-1} \sum_{i=0}^{p-1} \#\left\{v \in \mathbb{Z}_+^{n+1} \,\middle|\, \sum_{\substack{k=0 \\ k \neq j}}^{n} hv_k = 1 - ih,\ \sum_{\substack{k=0 \\ k \neq j}}^{n} x^k hv_k = x - ihx^j\right\} \\ &\quad = \binom{p+n}{n}^{-1} \sum_{i=0}^{p-1} \binom{p+n-i-1}{n-1} m_{h_i}(x - ihx^j \mid X \backslash \{x^j\}). \quad \blacksquare \end{aligned}$$

Next we state a convergence theorem for discrete B-splines.

THEOREM 3.2. *Let Y_h be a basis for $\tilde{\mathscr{L}}_h(X)$. Then there is a constant γ depending on X and s, such that*

$$|M(x \mid X) - |\det(Y_h)|^{-1} m_h(x \mid X)| \leqslant \gamma h,$$

for all $x \in \mathscr{S}_h(X)$.

Proof. Let Y_h be a basis for $\tilde{\mathscr{L}}_h(X)$ and $\tilde{Y}_h$ a basis for $\mathscr{L}_h(\tilde{X})$. Then it is easy to show that

$$|\det(Y_h)| = |\det(\tilde{Y}_h)|.$$

Moreover, it is known that

$$M(x \mid X) = n!\, T(\tilde{x} \mid \tilde{X}),$$

a continuous version of (3.11). The assertion follows now in a straightforward way from (3.11), the convergence result for discrete cone splines, and the simple fact that

$$\lim_{h \to 0} l(n, h) = n!. \quad \blacksquare$$

4. AN ALGORITHM FOR SUBDIVIDING SIMPLEX B-SPLINES

In this section we propose a method for subdividing simplex B-splines, based on Lemma 3.3.

LEMMA 4.1. *Let $n \geqslant s$. Consider the mesh $\mathscr{R}_h(\tilde{X})$, which is a restriction of $\mathscr{L}_h(\tilde{X})$, defined as*

$$\mathscr{R}_h(\tilde{X}) := \{\tilde{x} \in \mathscr{L}_h(\tilde{X}),\ \tilde{x}_{s+1} \in \{1 - nh, ..., 1 - h, 1\}\},$$

where $\tilde{x}_{s+1}$ is the $(s+1)$st coordinate of $\tilde{x}$. Then the control net $\{t_h(\tilde{x} \mid \tilde{X}),\ \tilde{x} \in \mathscr{R}_h(\tilde{X})\}$ is sufficient for subdividing the cone spline $T(\tilde{x} \mid \tilde{X})$ along the hyperplane $\tilde{x}_{s+1} = 1$.

Proof. Employing well known facts about cube and cone splines (cf. [7]) it can be shown that

$$T(\tilde{x} \mid \tilde{X}) = \sum_{\tilde{\alpha} \in \mathscr{L}_h(\tilde{X})} t_h(\tilde{\alpha} \mid \tilde{X})\, B(\tilde{x} - \tilde{\alpha} \mid \tilde{X}h), \qquad \tilde{x} \in \mathbb{R}^{s+1}.$$

Since the length of the support of $B(\cdot \mid \tilde{X}h)$ in the direction of $\tilde{x}_{s+1}$ is $h(n+1)$, the cube splines $B(\tilde{x} - \tilde{\alpha} \mid \tilde{X}h)$ do not vanish for $\tilde{x}_{s+1} = 1$ only if $\tilde{\alpha} \in \mathscr{R}_h(\tilde{X})$. Thus, $T(\tilde{x} \mid \tilde{X})$ is for $\tilde{x}_{s+1} = 1$ completely determined by the net $\{t_h(\tilde{x} \mid \tilde{X}),\ \tilde{x} \in \mathscr{R}_h(\tilde{X})\}$. $\blacksquare$

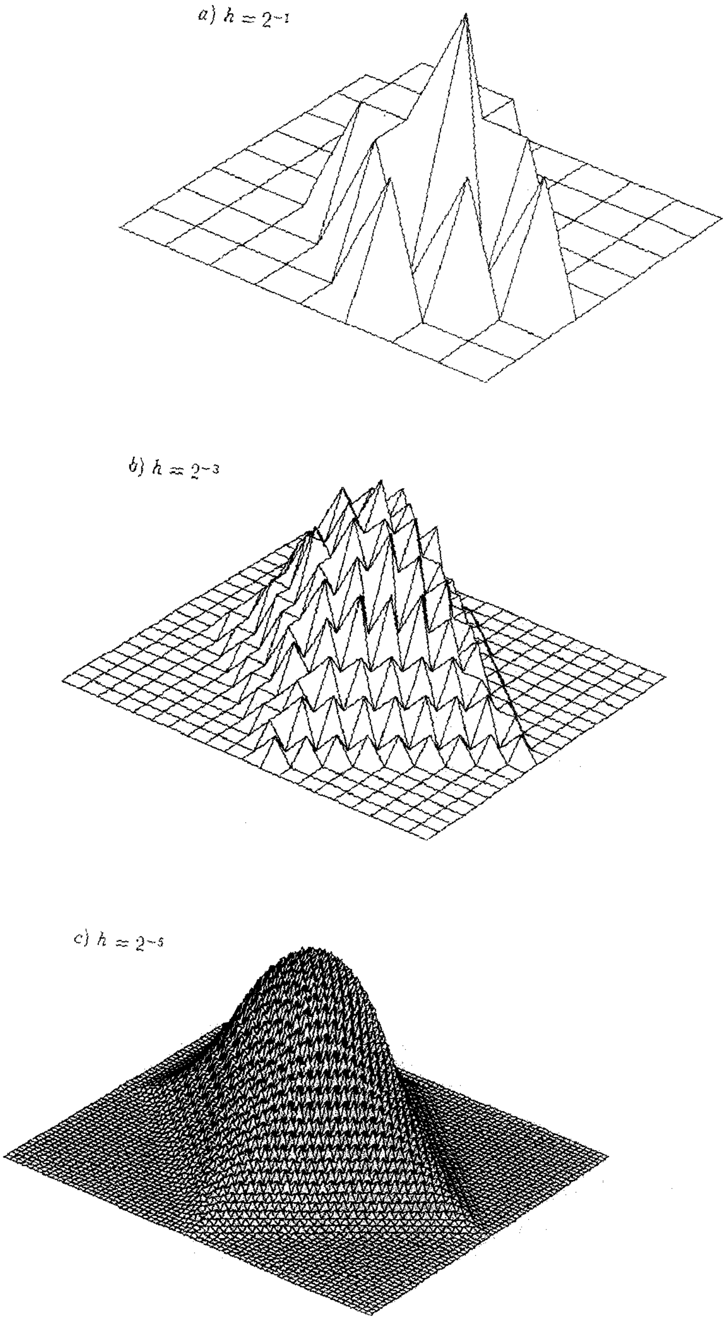

FIG. 4.1. A discrete quadratic B-spline $m_h(\cdot \mid X)$, $X = \{(1, 0), (0, 1), (2, 1), (1, 2), (2, 2)\}$.

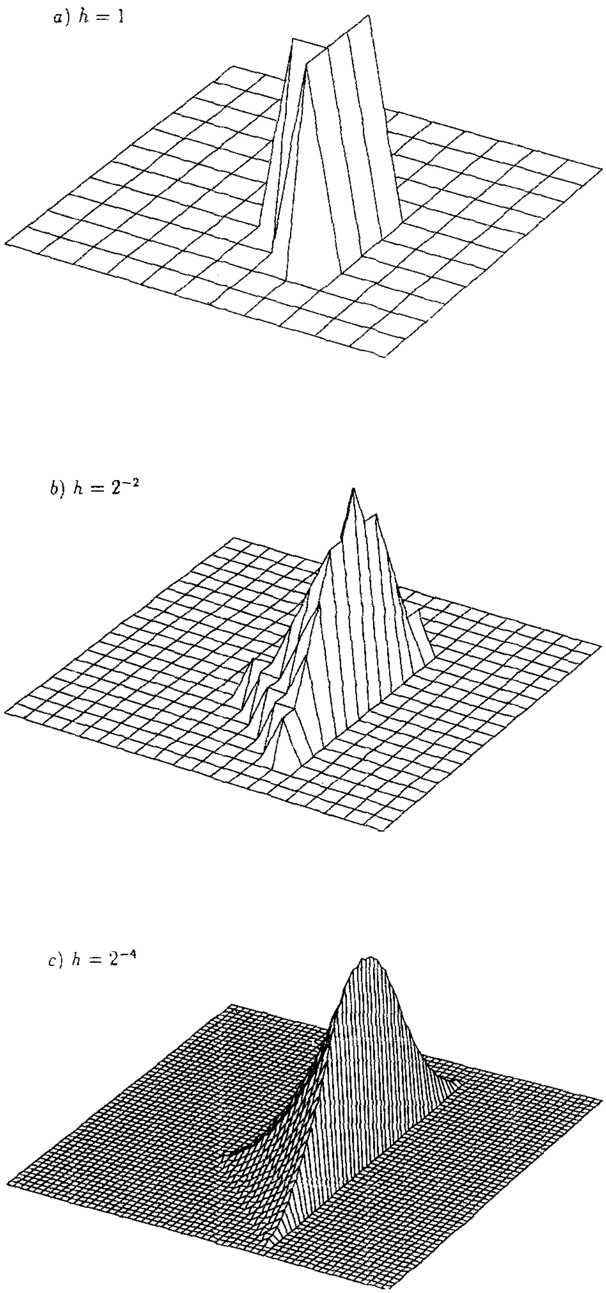

FIG. 4.2. A discrete quadratic B-spline $m_h(\cdot \mid X)$, $X = \{(0, 0), (0, 1), (0, 2), (0, 3), (2, 1)\}$.

Lemma 4.1 asserts that in order to have a sufficient information on the cone spline along the hyperplane $\tilde{x}_{s+1}=1$ it is necessary to keep track of $n+1$ "layers" of the subdivision net only, namely the layers corresponding to $\tilde{x}_{s+1}=1-nh, ..., \tilde{x}_{s+1}=1-h$, $\tilde{x}_{s+1}=1$. In the following let h_i, $i=1, 2$, be constants such that $h_i^{-1}=p_i \in \mathbb{N}$. Moreover, in analogy to Corollary 2.1, let $\bar{t}_{h_1}(\tilde{x} \mid \tilde{X})$ be a function defined for all $\tilde{x} \in \mathscr{L}_{h_1 h_2}(\tilde{X})$ as

$$\bar{t}_{h_1}(\tilde{x} \mid \tilde{X}) := \begin{cases} t_{h_1}(\tilde{x} \mid \tilde{X}), & \text{if } \tilde{x} \in \mathscr{L}_{h_1}(\tilde{X}) \\ 0, & \text{otherwise.} \end{cases}$$

Next we present an algorithm for refining the net $\{t_{h_1}(\tilde{x} \mid \tilde{X}), \tilde{x} \in \mathscr{R}_{h_1}(\tilde{X})\}$.

Algorithm. 1. Set $t^0_{h_1}(\tilde{x} \mid \tilde{X}) := \bar{t}_{h_1}(\tilde{x} \mid \tilde{X})$, for all $\tilde{x} \in \mathscr{L}_{h_1 h_2}(\tilde{X})$ such that, $\tilde{x}_{s+1} \in \{1-((n+1)\, p_2-1)\, h_1 h_2, ..., 1-h_1 h_2, 1\}$.

2. Set $t^i_{h_1}(\tilde{x} \mid \tilde{X}) := h_2 \sum_{j=0}^{p_2-1} t^{i-1}_{h_1}(\tilde{x}-jh_1 h_2 \tilde{x}^i \mid \tilde{X})$, for all $\tilde{x} \in \mathscr{L}_{h_1 h_2}(\tilde{X})$ such that, $\tilde{x}_{s+1} \in \{1-((n+1-i)\, p_2-1+i)\, h_1 h_2, ..., 1-h_1 h_2, 1\}$, $i=1, ..., n+1$.

3. $t_{h_1 h_2}(\tilde{x} \mid \tilde{X}) = t^{n+1}_{h_1}(\tilde{x} \mid \tilde{X})$, for all $\tilde{x} \in \mathscr{L}_{h_1 h_2}(\tilde{X})$ such that, $\tilde{x}_{s+1} \in \{1-nh_1 h_2, ..., 1-h_1 h_2, 1\}$.

4. $m_{h_1 h_2}(x \mid X) = l(n, h)\, t_{h_1 h_2}(\tilde{x} \mid \tilde{X})$, $x \in \mathscr{S}_h(X)$.

Observe, that in order to compute the discrete function $\bar{t}_{h_1}(\tilde{x} \mid \tilde{X})$ for those $\tilde{x}$ for which $\tilde{x}_{s+1} \in \{1-((n+1)\, p_2-1)\, h_1 h_2, ..., 1-h_1 h_2, 1\}$, only the values of the control net $\{t_{h_1}(\tilde{x} \mid \tilde{X}), \tilde{x} \in \mathscr{R}_{h_1}(\tilde{X})\}$ are needed, which is in accordance with Lemma 4.1. At the end of the algorithm we are left with the refined control net $\{t_{h_1 h_2}(\tilde{x} \mid \tilde{X}), \tilde{x} \in \mathscr{R}_{h_1 h_2}(\tilde{X})\}$, which represents a sufficient information for a next subdivision step. The fourth step of the algorithm can be performed in order to obtain the desired values of the discrete B-spline $m_{h_1 h_2}(x \mid X)$. It should be stressed, however, that this algorithm does not make it possible to calculate $m_{h_1 h_2}$ directly from $m_{h_1} = t_{h_1}(\tilde{x} \mid \tilde{X})$, $x \in \mathscr{S}_{h_1}(X)$ without knowing the other layers of $t_{h_1}(\tilde{x} \mid \tilde{X})$, namely for $\tilde{x}_{s+1} \in \{1-nh_1, ..., 1-h_1\}$.

Figures 4.1 and 4.2 present some examples of discrete B-splines for different values of h. The evaluation has been performed on the basis of the above algorithm.

5. Zig-zagging of the Control Net

Figures 4.1 and 4.2 confirm the convergence of discrete simplex splines to the corresponding continuous simplex splines. However, it can be seen that the subdivision nets are not visually pleasing in the sense that they contain undesired oscillations. We call this negative phenomenon of the control net *zig-zagging*. In the case of cube splines, this phenomenon is a consequence of well known facts about discrete cube splines. In particular, it follows from the results in [9] that a discrete cube spline is *not* a piecewise polynomial function unless the direction set X is *unimodular*. In the case when X is not unimodular, pieces of the discrete cube spline are known to contain oscillating terms satisfying certain partial difference equations.

Next we describe a possible remedy for removing the zig-zagging from the subdivision net. For the sake of brevity we restrict ourselves to the case of cube splines. The results apply in a straightforward manner to discrete cone splines and thus, by Theorem 3.1 and Lemma 3.3, also to discrete simplex splines. The idea is based on the identity

$$B_h(x \mid X) = h^{-s} \sum_{y \in \mathscr{L}_h(X)} b_h(y \mid X)\, B(h^{-1}(x-y) \mid X), \qquad x \in \mathscr{L}_h(X), \tag{5.1}$$

where $B_h(\cdot \mid X)$, $h^{-1} \in \mathbb{N}$, denotes the restriction of $B(\cdot \mid X)$ to $\mathscr{L}_h(X)$, i.e.,

$$B_h(x \mid X) := B(x \mid X), \qquad x \in \mathscr{L}_h(X).$$

This is an immediate consequence of the well known fact [3, 7]

$$B(x \mid X) = \sum_{y \in \mathscr{L}_h(X)} b_h(y \mid X)\, B(x-y \mid Xh), \qquad x \in \mathbb{R}^s,$$

and

$$B(x \mid Xh) = h^{-s} B(h^{-1}x \mid X), \qquad x \in \mathbb{R}^s.$$

Hence, from (5.1) it follows that exact values of $B(\cdot \mid X)$ on the fine grid $\mathscr{L}_h(X)$ can be obtained as a discrete convolution of the corresponding discrete cube spline $b_h(\cdot \mid X)$ on the *fine* grid with the "discretized" cube spline $B(\cdot \mid X)$ on the *coarsest* grid $\mathscr{L}(X)$. This suggests a refinement strategy in two stages. In the first stage the discrete cube splines (or a linear combination of discrete cube splines) are calculated as in Section 2. In the second stage, the refined control net is "smoothed out" by (5.1). In this way the subdivision gives rise to *exact* values of the cube spline surfaces. That means that the subdivision net is actually a piecewise linear interpolant of the limit surface and thus it is converging *quadratically* to it. Note, that in general the convergence of the control net corresponding to b_h is linear only.

The above method has also some drawbacks, however. First, exact values of the cube splines are needed. This is not a serious drawback, since only values on the coarsest grid must be evaluated, their number being, for typical cube splines of low degree, relatively small. These values can be computed, e.g., by a *local subdivision*, i.e., by subdivision which uses local refinement of the subdivision net, only. Moreover, they can be calculated and stored in advance, since they do not depend on the number of refinement steps and on the initial control net for $h=1$. A more serious shortcoming of the above method is an increased computational complexity of the subdivision, caused by the necessity of performing the convolution (5.1).

6. Discussion

In this paper the notion of discrete multivariate simplex splines has been introduced. We have shown that it fits naturally in the concepts of discrete cube and cone splines. Moreover, some basic properties of the discrete simplex splines have been derived, which are analogous to the ones in the continuous case.

Discrete univariate B-splines on a uniform grid have first been introduced by Schumaker [16], as piecewise polynomial discrete functions. It is worth noting that, in the univariate case, the splines defined here are not identical with Schumaker's discrete B-splines. The difference is that, in general, our discrete splines are *not* piecewise polynomial. A consequence of this fact is a phenomenon called zig-zagging of the refined control net, described in Section 5.

Acknowledgment

I thank the referee for a careful reading of the manuscript and his suggestions which improved the final version of the paper.

References

1. W. Boehm, Subdividing multivariate splines, *Comput. Aided Design* **15** (1983), 345–352.
2. A. S. Cavaretta and C. A. Michelli, The design of curves and surfaces by subdivision algorithms, *in* "Mathematical Methods in CAGD" (T. Lyche and L. L. Schumaker, Eds.), pp. 115–153, Academic Press, Boston, 1987.
3. E. T. Cohen, T. Lyche, and R. Riesenfeld, Discrete box splines and refinement algorithms, *Comput. Aided Geom. Design* **1** (1984), 131–148.
4. W. Dahmen, Multivariate B-splines—Recurrence relations and linear combinations of truncated powers, *in* "Multivariate Approximation Theory" (W. Schempp and K. Zeller, Eds.), pp. 64–82, Birkhäuser, Basel, 1979.
5. W. Dahmen, On multivariate B-splines, *SIAM J. Numer. Anal.* **17** (1980), 179–191.
6. W. Dahmen and C. A. Micchelli, Recent progress in multivariate splines, *in* "Approximation Theory, IV" (C. K. Chui, L. L. Schumaker, and J. D. Ward, Eds.), pp. 27–121, Academic Press, New York, 1983.
7. W. Dahmen and C. A. Micchelli, Subdivision algorithms for the generation of box spline surfaces, *Comput. Aided Geom. Design* **1** (1984), 115–129.

8. W. DAHMEN AND C. A. MICCHELLI, Line average algorithm: A method for the computer generation of smooth surfaces, *Comput. Aided Geom. Design* **2** (1985), 77–85.
9. W. DAHMEN AND C. A. MICCHELLI, On the solution of certain systems of partial differential equations and linear dependence of translates of box splines, *Trans. Amer. Math. Soc.* **292** (1985), 305–320.
10. W. DAHMEN AND C. A. MICCHELLI, Algebraic properties of discrete box splines, *Constr. Approx.* **3** (1987), 209–221.
11. W. DAHMEN AND C. A. MICCHELLI, The number of solutions to linear diophantine equations and multivariate splines, *Trans. Amer. Math. Soc.* **308** (1988), 509–532.
12. P. KOCHEVAR, An application of multivariate B-splines to computer aided geometric design, *Rocky Mountain J. Math.* **14** (1984), 159–175.
13. C. A. MICCHELLI, A constructive approach to Kergin interpolation in R^k: Multivariate B-splines and Lagrange interpolation, *Rocky Mountain J. Math.* **10** (1980), 485–497.
14. H. PRAUTZSCH, "Unterteilungsalgorithmen für Multivariate Splines," Ph.D. thesis, Technische Universität Braunschweig, 1984.
15. M. SABIN, Open questions in the application of multivariate B-splines, *in* "Mathematical Methods in CAGD" (T. Lyche and L. L. Schumaker, Eds.), pp. 529–537, Academic Press, Boston, 1989.
16. L. L. SCHUMAKER, Constructive aspects of discrete polynomial spline functions, *in* "Approximation Theory" (G. G. Lorentz, Ed.), pp. 469–476, Academic Press, New York, 1973.